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ON RELATIONAL DOMAINS, THE ALGEBRA OF RELATIONS, AND RELATIONAL-TERM LOGIC

"Mathematics is tricks."
Henry Pollock
"No, mathematics is thought."
Abner Shimony

Well, perphaps the appropriate addendum to the foregoing dialogue is to remark that mathematics is thought about tricks and the reasons for them - about the thousands upon thousands of minutiae that make up mathematical language when stripped to its ultimate notions, in the manner, say, of *Principia Mathematica* and allied systems. Of particular importance here are the technicalia of quantification theory, as formidable a bag of tricks as has e'er been thought up. That this is the case becomes especially evident when that theory is looked at in the light of other theories that purport to take its place in some fashion or other - such as relational algebra, combinatory logic, or predicate-functor logic.

In a previous paper[1] a preliminary sketch of a pure algebra of n-adic relations - call it 'RA' - was put forward, 'pure' in the sense that no set-theory was made use of in any way at either the object or metalinguistic level. This kind of formulation is in marked contrast with previous formulations. A purported interpretation of the theory was given in the theory of virtual classes and relations as based on the ordinary first-order theory of quantification.[2] No axiomatization of RA was given, however, and only a feeble attempt was made to show how quantification theory itself is contained within it in a kind of notational disguise. Let us attempt now, in the present paper, to formulate a simpler kind of theory as a surrogate for it.

Let us recall, first, some basic features of RA itself and of the theory of virtual relations on which it is based. A supply of primitive non-logical relational constants, each of specified finite degree, is presupposed. The algebra RA is thus an *applied* one in this sense (but pure of course in presupposing no set-theory). Boolean operations of forming *sums*, *products*, and *negations* of n-adic relations are also presupposed, as well as the *universal* and *null* relations of each degree, n, where $n \geqq 1$. Where $n = 1$, we speak of a *monadic* relation, as is more or less customary. In addition, an operation for the *Cartesian product* of two relations is needed, as well as a notation for all the *domains* of a relation. The notion of domain here is the generalized one appropriate for n-adic relations. These latter two notions seem not to have been studied very much for their own sake and in a generalized form, but are of the highest interest philosophically.

These various notions may be characterized in terms of virtual relations as follows. Let

$$\{x_1 \ldots x_n \ni \text{--}x_1\text{--} \ldots \text{--}x_n\text{--}\}$$

be the virtual relation among $x_1, \ldots, x_n$ where '$\text{--}x_1\text{--} \ldots \text{--}x_n\text{--}$', some formula of quantification theory constructed in terms of the non-logical relational primitives, holds. We can then let

$\ulcorner(R_n \cup S_n)_n\urcorner$ abbreviate $\ulcorner\{x_1 \ldots x_n \ni (R_n x_1 \ldots x_n \vee S_n x_1 \ldots x_n)\}\urcorner$,

$\ulcorner(R_n \cap S_n)_n\urcorner$ abbreviate $\ulcorner\{x_1 \ldots x_n \ni (R_n x_1 \ldots x_n \cdot S_n x_1 \ldots x_n)\}\urcorner$,

$\ulcorner -R_n\urcorner$ abbreviate $\ulcorner\{x_1 \ldots x_n \ni \sim R_n x_1 \ldots x_n\}\urcorner$,

'V_n' abbreviate '$\{x_1 \ldots x_n \ni (x_1 = x_1 \cdot x_2 = x_2 \cdot \ldots \cdot x_n = x_n)\}$'

and

'Λ' abbreviate '$-V_n$'.

These abbreviations, within the theory of virtual classes and relations, introduce the respective Boolean notions. The Cartesian product may be defined by letting

$$\ulcorner(R_n \times S_m)_{(n+m)}\urcorner \text{ abbreviate } \ulcorner\{x_1 \ldots x_n y_1 \ldots y_m \ni R_n x_1 \ldots x_n \cdot S_m y_1 \ldots y_m)\}\urcorner.$$

Note that the Boolean operations are *homeoadic* in the sense that the result of the operation is of the same degree n as that of the operands, and that the two operands (of the sums and products) are of the same degree n. The degree of the Cartesian product, however, is *cumulative* in the sense that the degree $(n+m)$ of the result of the operation is the (arithmetical) sum of the degrees n and m of the two operands. The notion of the identity of individuals used here may be taken either as a primitive (as in quantification theory with identity) or else defined, say, in the manner of Hilbert and Bernays.[3] Let it be represented here by 'Id_2', no matter how introduced.

The explanations in the foregoing paragraph are of course merely heuristic, and play no role in RA itself. They merely help us to see how its primitives are to be interpreted. And similarly now for the generalized notion of the domain of a relation.

For a dyadic relation R_2, the first domain is merely

(1) $$\{x \ni (Ey)R_2\, xy\},$$

and the second or converse domain is

(2) $$\{y \ni (Ex)R_2\, xy\}$$

For a triadic relation R_3, we must provide not only for

(3) $$\{x \ni (Ey)(Ez)R_3\, xyz\},$$

(4) $$\{y \ni (Ex)(Ez)R_3\, xyz\},$$

and

(5) $$\{\underline{z} \ni (E\underline{x})(E\underline{y})R_3\ \underline{xyz}\}\ ,$$

but also for

(6) $$\{\underline{xy} \ni (E\underline{z})R_3\ \underline{xyz}\}\ ,$$

(7) $$\{\underline{xz} \ni (E\underline{y})R_3\ \underline{xyz}\}$$

and

(8) $$\{\underline{yz} \ni (E\underline{x})R_3\ \underline{xyz}\}\ .$$

The first of these we may think of as the domain, the third as the converse domain, and the second as the middle domain. The other three are *relational* domains, but no obvious terminology for them is available. In the case of tetradic (or quadratic) relations, the situation is still more complicated. To provide for all these, and in general for all the domains including the relational ones, let us let

$$\ulcorner(D_{\underline{i},\ldots,\underline{h}}\ \underset{\approx}{R}_{\underline{n}})_{\underline{j}}\urcorner \text{ abbreviate } \ulcorner\{\underline{x}_{\underline{i}}\ldots\underline{x}_{\underline{h}} \ni (E\underline{x}_{\underline{k}})\ \ldots\ (E\underline{x}_{\underline{m}})\underset{\approx}{R}_{\underline{n}}\ \underline{x}_1\ \ldots\ \underline{x}_{\underline{n}}\}\urcorner,$$

where $\underline{i}$, ..., $\underline{h}$, $\underline{k}$, ..., $\underline{m}$ are any $\underline{n}$ distinct positive integers each $\leqq$ $\underline{n}$, $\underline{i}$, ..., $\underline{h}$ are just $\underline{j}$ in number ($\underline{j} \geqq 1$), and $\underline{i}$, ..., $\underline{h}$ and $\underline{k}$, ..., $\underline{m}$ are taken in any order of magnitude, for $\underline{n} \geqq 2$.

We note now how (1)-(8) may be represented or symbolized in terms of 'D', (1) in fact is merely $(D_1R_2)_1$ and (2) is $(D_2R_2)_1$. (3) is $(D_1R_3)_1$, (4) is $(D_2R_3)_1$, (5) is $(D_3R_3)_1$. (6), (7), and (8) are themselves dyadic relations represented by '$(D_{1,2}R_3)_2$', '$(D_{1,3}R_3)_2$' and '$(D_{2,3}R_3)_2$' respectively. And so on for relations of higher degree and their domains, relational and other.
Let us note that expressions such as '$(D_{1,2,3}R_3)_3$' are also significant, so that a relation is allowed to be one of its own domains. But so is '$(D_{1,3,2}R_3)_3$' significant, and '$(D_{2,1,3}R_3)_3$', '$(D_{2,3,1}R_3)_3$', '$(D_{3,1,2}R_3)_3$', and '$(D_{3,2,1}R_3)_3$'. Each of these represents one of the *converses* of the triadic relation R_3 just as '$(D_{2,1}R_2)_2$' represents the converse of R_2. And similarly for relations of higher degree, so that the general theory of converses is included in the general theory

of domains.

Note, by way of a few further examples, that

(9) $$\ulcorner\{\underline{x} \ni (\underline{y})\mathrm{R}_2\ \underline{xy}\}\urcorner$$

is represented here by

$$\ulcorner-(\mathrm{D}_1\ -\mathrm{R}_2)\urcorner,$$

(10) $$\ulcorner\{\underline{x} \ni (\mathrm{E}\underline{y})(\underline{z})(\underline{w})(\mathrm{E}\underline{u})(\underline{v})\mathrm{R}_6\ \underline{xyzwuv}\}\urcorner$$

by

$$\ulcorner(\mathrm{D}_1-(\mathrm{D}_{1,2}-(\mathrm{D}_{1,2,3,4}-(\mathrm{D}_{1,2,3,4,5,}-\mathrm{R}_6)_5)_4)_2)_1)\urcorner,$$

(11) $$\ulcorner\{\underline{xwyz} \ni \mathrm{R}_4\ \underline{xyzw}\}\urcorner$$

by $$\ulcorner(\mathrm{D}_{1,4,2,3}\mathrm{R}_4)_4\urcorner,$$

(12) $$\ulcorner\{\underline{xy} \ni (\mathrm{E}\underline{z})(\mathrm{E}\underline{w})(\underline{u})(\mathrm{E}\underline{v})\mathrm{R}_6\ \underline{xvyuwz}\}\urcorner$$

by

$$\ulcorner(\mathrm{D}_{1,2}-(\mathrm{D}_{1,2,3,4}-(\mathrm{D}_{1,2,3,4,5}(\mathrm{D}_{1,6,2,5,4,3}\mathrm{R}_6)_6)_5)_4)_2\urcorner.$$

In the formulation of RA, all talk of virtual classes and relations is, of course, dropped, the effect of such being achieved by use of the Boolean notions, Cartesian products, and domains. In an algebra there is always a primitive '=' for identity, so that $\ulcorner \mathrm{R}_n = \mathrm{S}_n \urcorner$ is the only kind of atomic formula admitted, but with '$\sim$' and 'v', say, available as truth-functional primitives to provide for molecular formulae. The basic principles of RA -- and, ultimately, the axioms -- must provide for the truth-functions, for identity as between $\underline{n}$-adic relations, for the Boolean notions, and for Cartesian products and domains. A list of some of the principles needed was put forward in the previous paper. Not all of these are required as axioms, although some additional ones are.

In the original presentation of RA in *Mind, Modality, Meaning, and Method*, the sentence

(13)

$$\text{'}(E\underline{x})(\underline{y})(E\underline{u})(\underline{v})T_4\ \underline{xyuv}\text{'}$$

was said to be expressible by

(14) $$\text{'}\sim -(D_{1,3} - T_4)_2 = \Lambda_2\text{'}\ .$$

But unfortunately this is not correct. This latter states, in virtual-relation terms, that

$$\sim\{\underline{xu} \ni (E\underline{y})(E\underline{v}) - T_4\ \underline{xyuv}\} = \Lambda_1,$$

or, equivalently, that

(15) $$(E\underline{x})(E\underline{u})(\underline{y})(\underline{v}) - T_4\ \underline{xyuv}\ .$$

But of course (15) is not equivalent to (13), although it logically implies it. The form for (13) should be rather

$$\text{'}-(D_1-(D_{1,2}-(D_{1,2,3}-T_4)_3)_2)_1 = \Lambda_1\text{'},$$

as may easily be verified. And, similarly, the Axiom of Pairs of set-theory has the form

$$\text{'}(D_{1,2}-(D_{1,2,3}-(((D_{1,4,2,3}(E_2 \times V_2)_4)_4) \cap ((Id_2 \times V_2)_4 \cup (D_{1,3,2,4}$$

$$(Id_2 \times V_2)_4)_4)_4)_4 \cup (-(D_{1,4,2,3}(E_2 \times V_2)_4)_4 \cap (-(Id_2 \times V_2)_4 \cap$$

$$-(D_{1,3,2,4}(Id_2 \times V_2)_4)_4)_4)_4)_4)_3)_2 = V_2\text{'},$$

and the given instances of the *Aussonderungsschema* the form

$$\ulcorner(D_1-(D_{1,2}-(((E_2 \times V_1)_3 \cap ((\underline{\underline{R}}_1 \times V_2)_3 \cap (D_{1,3,2}(E_2 \times V_1)_3)_3)_3)_3$$

$$\cup (-(E_2 \times V_1)_3 \cap (-(\underline{\underline{R}}_1 \times V_2)_3 \cup -(D_{1,3,2}(E_2 \times V_1)_3)_3)_3)_3)_3)_2)_1 = V_1\urcorner.$$

Given a relation $R_{\underline{n}}$, for $\underline{n} \geqq 2$, the totality of its converses together with all its domains constitutes a family of closely *affiliated* relations. All of the affiliates are constructed from the given one and are recognized as relations along with it. In some sense, they are "given" along with the original one, merely awaiting a proper

notation, as it were. Once we have such a notation, RA is seen to be merely an extension of an essentially Boolean theory as augmented with Cartesian products. The admission of the relations affiliated with a given relation thus seems not only natural, but a small price to pay for the algebraic richness forthcoming as a result. Further, RA itself may now be seen to wholly Booleanized, so to speak, all erstwhile relational notions -- intrisically relational or non-Boolean ones -- being now assimilated in the theory of domains.
It should be observed that '=' for identity as between $\underline{n}$-adic relations in RA is a very strong primitive indeed, and the two axiom-schemata governing it are very strong principles. These are that

$$\vdash R_{\underline{n}} = R_{\underline{n}} \ ,$$

and

$\vdash R_{\underline{n}} = S_{\underline{n}} \supset {-}{-}R_{\underline{n}}{-}{-} = {-}{-}S_{\underline{n}}{-}{-}$, where $\ulcorner{-}{-}S_{\underline{n}}{-}{-}\urcorner$ differs from the term $\ulcorner{-}{-}R_{\underline{n}}{-}{-}\urcorner$ only in containing occurrences of $S_{\underline{n}}$ in one or more places where there are occurrences of $R_{\underline{n}}$ in $\ulcorner{-}{-}R_{\underline{n}}{-}{-}\urcorner$.

It is interesting to observe that the effect of having identity may be achieved in a much more economical way. The result will no longer be an algebra, however, but a special kind of logic of relations.
We should observe also that the truth-functional connectives may be eliminated as primitives of RA without loss. Recall the principles that

$$\vdash \sim R_{\underline{n}} = V_{\underline{n}} \equiv (D_1(V_1 \times -R_{\underline{n}})_{(\underline{n}+1)})_1 = V_1 \ ,$$

$$\vdash (R_{\underline{n}} = V_{\underline{n}} \ . \ S_{\underline{n}} = V_{\underline{n}}) \equiv (R_{\underline{n}} \cap S_{\underline{n}})_{\underline{n}} = V_{\underline{n}} \ ,$$

and

$$\vdash (\sim R_{\underline{n}} = V_{\underline{n}} \vee \sim S_{\underline{n}} = V_{\underline{n}}) \equiv (D_1(V_1 \times -(R_{\underline{n}} \cap S_{\underline{n}})_{\underline{n}})_{(\underline{n}+1)})_1 = V_1 \ ,$$

or, better,

$$\vdash (R_{\underline{n}} = V_{\underline{n}} \vee S_{\underline{n}} = V_{\underline{n}}) \equiv (-(D_1(V_1 \times -R_{\underline{n}})\ \ )_{(\underline{n}+1)})_1 \cup -(D_1(V_1 \times -S_{\underline{n}})\ \ )_{(\underline{n}+1)})_1)_1 = V_1 .$$

Whenever, in RA, we wish to use a truth-functional connective, we may eliminate it in view of these equivalences.

It should be observed also that the only atomic formulae we need consider are of the form $\ulcorner T_{\underline{n}} = V_{\underline{n}} \urcorner$ in view of the Boolean principle that

$$\vdash R_{\underline{n}} = S_{\underline{n}} \equiv ((-R_{\underline{n}} \cup S_{\underline{n}})_{\underline{n}} \cap (R_{\underline{n}} \cup -S_{\underline{n}})_{\underline{n}})_{\underline{n}} = V_{\underline{n}} .$$

It is true that '=' occurs in this equivalence on the right-hand side. In place of $\ulcorner T_{\underline{n}} = V_{\underline{n}} \urcorner$, let us now write merely

$$\ulcorner V_{\underline{n}}\, T_{\underline{n}} \urcorner,$$

to the effect that such and such and $\underline{n}$-adic relation is *universal* in the sense appropriate to $\underline{n}$ -adic relations. No properties of identity as such need then be postulated, as we shall see in a moment. Actually, we can simplify RA in another respect also, by using primitively only Cartesian products of the form $\ulcorner (R_{\underline{n}} \times V_1)_{(\underline{n}+1)} \urcorner$, where V_1 is the universal monadic relation.

We shall see how the full effect of having Cartesian products, may be achieved in this way.

Let us go on, now, to formulate the essentials of RA, but without using identity or the truth-functions. As primitives, let us take '-' for Boolean negation, '∪' for Boolean sums, '×' for the restricted Cartesian products, and 'D' with suitable subscripts for relational domains. Erstwhile individual constants are to be handled predicatively, so that 'Soc', for example, may represent the predicate 'Socratizes'. At least one such primitive is presupposed, say 'Soc' itself.

The following recursive specification is of the general notion of being a *relational term* of degree $\underline{n}$ where $\underline{n} \geqq 1$.

1. 'Soc' (and any other primitive for an erstwhile proper name) is a relational term of degree 1.

2. If R_n is a relational term of degree n, then so is $\ulcorner -R_n \urcorner$.

3. If R_n and S_n are relational terms of degree n, so is $\ulcorner (R_n \cup S_n)_n \urcorner$.

4. If R_n is a relational term of degree n, $\ulcorner (R_n \times (Soc \cup -Soc)_1)_{(n+1)} \urcorner$ is a relational term of degree $(n+1)$.

5. If $R_{(n+m)}$ is a relational term of degree $(n+m)$, $\ulcorner (D_{i,\ldots,j} R_{(n+m)})_n \urcorner$ is a relational term of degree n, where (as needed).

To facilitate the notation, we may let

D1. 'V_1' abbreviate '$(Soc \cup -Soc)_1$',

and then

D2a. 'V_2' abbreviate '$(V_1 \times V_1)_2$',

D2b. 'V_3' abbreviate '$(V_2 \times V_1)_2$',

and so on. In addition we may let

D3a. $\ulcorner (R_n \times V_2)_{(n+2)} \urcorner$ abbreviate $\ulcorner ((R_n \times V_1)_{(n+1)} \times V_1)_{(n+2)} \urcorner$,

D3b. $\ulcorner (R_n \times V_3)_{(n+3)} \urcorner$ abbreviate $\ulcorner ((R_n \times V_2)_{(n+2)} \times V_1)_{(n+3)} \urcorner$,

and so on.

D4. $\ulcorner (V_n \times R_m)_{(n+m)} \urcorner$ abbreviates $\ulcorner (D_{(m+1),\ldots,(m+n),1,\ldots,m} (R_m \times V_n)_{(n+m)})_{(n+m)} \urcorner$,

D5. $\ulcorner (R_n \times S_m)_{(n+m)} \urcorner$ abbreviates $\ulcorner ((R_n \times V_m)_{(n+m)} \cap (V_n \times$

$S_m)_{(n+m)})_{(n+m)}$⌝.

In this way we can achieve the full effect of having Cartesian products.

By a *formula* let us now understand any expression of the form ⌜$V_n\ R_n$⌝ where R_n is a relational term of degree n. The formulae are to be understood as saying that such and such a relation of degree n has the kind of universality appropriate to n-adic relations. The only expressions allowed in this kind of logic are the relational terms. Let us therefore call it 'relational-term logic', or for short, 'RTL'.

A few further abbreviations are useful. We may let

D6. ⌜$(R_n \cap S_n)_n$⌝ be short for ⌜$-(-R_n \cup -S_n)_n$⌝,

D7a. 'Λ_1' for '$-V_1$',

D7b. 'Λ_2' for '$(\Lambda_1 \times \Lambda_1)_2$',

and so on,

D8. ⌜$(R_n \supset S_n)_n$⌝ for ⌜$(-R_n \cup S_n)_n$⌝,

and

D9. ⌜$(R_n \equiv S_n)_n$⌝ for ⌜$((R_n \supset S_n)_n \cap (S_n \supset R_n)_n)_n$⌝.

D3 and D4 are definitions of a familiar kind, D5 and D6 less so. D5 introduces the notion of the Boolean *implexion* of n-adic relations, and D6 that of the Boolean *identification* of such. In terms of these the notions of *inclusion* and *identity* of n-adic relations may be introduced, by letting

D10. ⌜$R_n \subset S_n$⌝ abbreviate ⌜$V_n\ (R_n \supset S_n)_n$⌝,

and

<u>D11.</u> $\ulcorner R_{\underline{n}} = S_{\underline{n}} \urcorner$ abbreviate $\ulcorner V_{\underline{n}} (R_{\underline{n}} \sqsubset\!\sqsupset S_{\underline{n}})_{\underline{n}} \urcorner$.

Let us go on now to list some fundamental principles for RTL as follows. We first have some Boolean ones.

<u>Pr1.</u> $\vdash (R_{\underline{n}} \cup R_{\underline{n}})_{\underline{n}} \subset R_{\underline{n}}$,

<u>Pr2.</u> $\vdash R_{\underline{n}} \subset (R_{\underline{n}} \cup S_{\underline{n}})_{\underline{n}}$,

<u>Pr3.</u> $\vdash (R_{\underline{n}} \cup S_{\underline{n}})_{\underline{n}} \subset (S_{\underline{n}} \cup R_{\underline{n}})_{\underline{n}}$,

<u>Pr4.</u> $\vdash (R_{\underline{n}} \supset S_{\underline{n}})_{\underline{n}} \subset ((T_{\underline{n}} \cup R_{\underline{n}})_{\underline{n}} \supset (T_{\underline{n}} \cup S_{\underline{n}})_{\underline{n}})_{\underline{n}}$,

<u>MP.</u> If $\vdash V_{\underline{n}} R_{\underline{n}}$ and $\vdash R_{\underline{n}} \subset S_{\underline{n}}$, then $\vdash V_{\underline{n}} S_{\underline{n}}$.

These principles will be recognized as sufficient for providing for a Boolean algebra of $\underline{n}$-adic relations -- together with an existence assumption that will be given in a moment. <u>MP</u> is of course the adaptation of *modus ponens* needed for $\underline{n}$-adic relations.

The following principles govern the restricted Cartesian products and the relational domains.

<u>Pr5.</u> $\vdash (-R_{\underline{n}} \times V_1)_{(\underline{n}+1)} = -(R_{\underline{n}} \times V_1)_{(\underline{n}+1)}$,

<u>Pr6.</u> $\vdash ((R_{\underline{n}} \cup S_{\underline{n}})_{\underline{n}} \times V_1)_{(\underline{n}+1)} = ((R_{\underline{n}} \times V_1)_{(\underline{n}+1)} \cup (S_{\underline{n}} \times V_1)_{(\underline{n}+1)})_{(\underline{n}+1)}$,

<u>Pr7a.</u> $\vdash (D_{1,\ldots,\underline{n}}(R_{\underline{n}} \times V_1)_{(\underline{n}+1)})_{\underline{n}} = R_{\underline{n}}$,

<u>Pr7b.</u> $\vdash -(D_{1,\ldots,\underline{n}}(-R_{\underline{n}} \times V_1)_{(\underline{n}+1)})_{\underline{n}} = R_{\underline{n}}$,

<u>Pr8.</u> $\vdash -(D_{1,\ldots,\underline{n}} -(R_{(\underline{n}+1)} \supset S_{(\underline{n}+1)})_{(\underline{n}+1)})_{\underline{n}} \subset (-(D_{1,\ldots,\underline{n}} -R_{(\underline{n}+1)})_{\underline{n}} \supset -(D_{1,\ldots,\underline{n}} -S_{(\underline{n}+1)})_{\underline{n}})_{\underline{n}}$,

__Pr9.__ $\vdash (D_{1,\ldots,(\underline{n}-1),(\underline{n}+1)}{}^{-}(\underline{\underline{R}}_{\underline{n}} \times V_1)_{(\underline{n}+1)})_{\underline{n}} \subset -(D_{1,\ldots,(\underline{n}-1),}$

$(\underline{n}+1),\underline{n}^{-}(\underline{\underline{R}}_{\underline{n}} \times V_1)_{(\underline{n}+1)})_{(\underline{n}+1)})_{\underline{n}}$,

__Pr10.__ $\vdash -(D_{1,\ldots,(\underline{n}-1),(\underline{n}+1)}{}^{-}(\underline{\underline{R}}_{\underline{n}} \times \;\; {}_1)_{(\underline{n}+1)})_{\underline{n}} \quad -(D_{1,\ldots\underline{n}}$

$(D_{1,\ldots,(\underline{n}-1),(\underline{n}+1),\;\underline{n}}{}^{-}(\underline{\underline{R}}_{\underline{n}} \times V_1)_{(\underline{n}+1)})_{(\underline{n}+1)})_{\underline{n}}$, where α_1 is any primitive for an individual constant,

__Gen.__ If $\vdash V_{\underline{n}}\, \underline{\underline{R}}_{\underline{n}}$ then $V_1 - (D_{(\underline{n}+1)}(-\underline{\underline{R}}_{\underline{n}} \times V_1)_{(\underline{n}+1)})_1$.

We have also some principles governing converses, as follows.

__Pr11.__ $\vdash (D_{\underline{i},\ldots,\underline{j}}\underline{\underline{R}}_{\underline{n}} = -(D_{\underline{i},\ldots,\underline{j}}{}^{-}\underline{\underline{R}}_{\underline{n}})_{\underline{n}}$, where $\underline{i},\ldots,\underline{j}$ are any $\underline{n}$ distinct positive integers, each $\leqq \underline{n}$,

__Pr12.__ $\vdash (D_{\underline{i},\ldots,\underline{j}}(\underline{\underline{R}}_{\underline{n}} \cup \underline{\underline{S}}_{\underline{n}})_{\underline{n}})_{\underline{n}} \subset ((D_{\underline{i},\ldots,\underline{j}}\underline{\underline{R}}_{\underline{n}})_{\underline{n}} \cup (D_{\underline{i},\ldots,\underline{j}}\underline{\underline{S}}_{\underline{n}})_{\underline{n}})_{\underline{n}}$, where (etc.),

__Pr13.__ $\vdash (\underline{\underline{R}}_{\underline{n}} \supset \underline{\underline{S}}_{\underline{n}})_{\underline{n}} \subset ((D_{\underline{i},\ldots,\underline{j}}\underline{\underline{R}}_{\underline{n}})_{\underline{n}} \supset (D_{\underline{i},\ldots,\underline{j}}\underline{\underline{S}}_{\underline{n}})_{\underline{n}})_{\underline{n}}$,

__Pr14a.__ $\vdash (D_{2,1}(D_{2,1}\underline{\underline{R}}_2)_2)_2 = \underline{\underline{R}}_2$,

__Pr14b.__ $\vdash (D_{1,2}(D_{1,2}\underline{\underline{R}}_2)_2)_2 = \underline{\underline{R}}_2$,

__Pr14c.__ $\vdash (D_{1,3,2}(D_{1,3,2}\underline{\underline{R}}_3)_3)_3 = \underline{\underline{R}}_3$,

__Pr14d.__ $\vdash (D_{2,1,3}(D_{2,1,3}\underline{\underline{R}}_3)_3)_3 = \underline{\underline{R}}_3$,

__Pr14e.__ $\vdash (D_{3,1,2}(D_{2,3,1}\underline{\underline{R}}_3)_3)_3 = \underline{\underline{R}}_3$,

__Pr14f.__ $\vdash (D_{2,3,1}(D_{3,1,2}\underline{\underline{R}}_3)_3)_3 = \underline{\underline{R}}_3$,

$\underline{\text{Pr14g.}}$ $\vdash (D_{3,2,1}(D_{3,2,1}\underset{=}{R}_3)_3)_3 = \underset{=}{R}_3$,

and so on for relations of higher degree.
Finally, we need a general principle concerning the uniqueness and existence of the monadic relation Soc and other relations representing erstwhile individual constants.

$\underline{\text{Pr15.}}$ $\vdash \alpha = -(D_1 \quad -((\alpha \times V_1)_2 \cap ((D_{2,1} \, (\alpha \times V_1)_2 \supset Id_2)_2)_2)_1$,

where α is any primitive for an individual constant (and Id_2 is the relation of identity of individuals).

These principles are by no means exhaustive but are given as mere samples of principles that would be needed in any full axiomatization.
We need not tarry here with the deductive development of RTL. Much work yet remains to be done to achieve the minimum axioms sufficing for the theory -- to say nothing of stating precisely what it means to say this and then proving that it obtains. To do this last, in a satisfactory way, would require a semantical metalanguage, with its attendant axioms and rules, in which the proof could be carried out. A very considerable effort would be required to formulate such a metalanguage for such a proof, and many unforseen technical difficulties would no doubt arise. The same of course is to be said for all-called "proofs" of consistency, completeness and independence.[5]

An objection that might be raised against the notation for domains used here is that it makes abundant use of numerical subscripts and therefore the theories here presupposes arithmetic. But no. The numerical subscripts are used as mere notational markers, and their use here is as mere syntactical abbreviation, as would become pellucid if the syntax were spelled out in detail in terms of concatenation and shape-descriptions.[6] Also it might be objected that although variables such as '$\underline{x}$', '$\underline{y}$', and so on, are not needed in RA or RTL, abundant use is made of '$\underset{=}{R}_n$', and so on, as syntactical variables. True, but if the syntax, again, were fully spelled out, it in turn could be viewed as an applied relational-term logic without syntactical variables.

Let us note how relational domains are of use in giving logical forms of certain natural sentences. The first person ever to make use of them, or even to be aware of their existence, was probably C.S. Peirce. Let us turn, therefore, to an important paper of his of 1870.[7] This was long before he became aware of the quantifiers, and it is interesting to note that he felt the need of generalized domains to handle certain natural noun-phrases.

Consider, by way merely of an example, the sentence that "every lover of somebody who is a servant to nothing but a woman stands to nothing but women in the relation of lover of nothing but a servant of them." The two noun-phrases needed here may readily be symbolized in virtual-relation terms, and hence in RA or RTL. Let 'L_2' and 'S_2' stand for the dyadic relations of loving and being a servant of, respectively, and let 'W_1' stand for the class of women. Then the required noun-phrases, in quantificational terms, are

$$'\{\underline{x} \ni (E\underline{y})((\underline{z})(S_2\,\underline{yz} \supset W_1\underline{z}) \;.\; L_2\,\underline{xy})\}' ,$$

and

$$'\{\underline{x} \ni (\underline{y})((\underline{z})(L_2\,\underline{xz} \supset S_2\,\underline{zy}) \supset W_1\,\underline{y})\}'.$$

In the symbolism of RTL, these become

$$'(D_1 - (D_{1,2} - (((V_1 \times S_2)_3 \supset (V_2 \times W_1)_3)_3 \cap (L_2 \times V_1)_3)_3)_2)_1'$$

and

$$'-(D_1 - (D_{1,2}(D_{1,3,2}(((L_2 \times V_1)_3 \supset (V_1 \times S_2)_3)_3 \supset (V_2 \times W_1)_3)_3)_3)_2)_1'.$$

Another example from Peirce is the sentence "every lover of somebody who is servant to nothing but a woman stands to nothing but woman in the relation of lover of nothing but a servant of them." The two noun-phrases here, in the notation of RTL, are, respectively,

$$'-(D_1 - (D_{1,2}(D_{1,3,2}(((L_2 \times V_1)_3 \supset (V_1 \times S_2)_3)_3 \cap (V_1 \times W_1 \times V_1)_3)_3)_3)_2)_1,$$

and

$$'-(D_1-(D_{1,2}((L_2 \times V_1)_3 \supset ((V_1 \times S_2)_3 \cap (V_2 \times W_1)_3)_3)_3)_2)_1' .$$

Peirce's two sentences are then formed by placing the inclusion-sign '$\subset$' between the two relevant phrases. The results, incidentally, are logically true statements, as Peirce in effect observes. To the uninitiate, these examples may seem somewhat far-fetched. These and similar noun-phrases, however, abound in natural language. It is, therefore, essential that a method for handling them be available in any general theory of relations adequate for the normal tasks expected of it. It is interesting to observe also that it was for a notation for relational domains of any complexity that Peirce was struggling, in his early papers on the algebra and logic of relations, before coming to terms with the quantifiers in his papers of 1883 and 1885.[8]

Let us go on now to show how RTL is but a step or so removed from a kind of *combinatory* logic of relations, one in which the combinators or operators apply to relations as the result.[9] We shall need just six such combinators, namely,

S, N, P, C, I, and E.

S will provide for Boolean sums on $\underline{n}$-adic relations, N for negations, P for the restricted Cartesian products, C for *conversion*, I for *inversion*, and E for handling the construction of domains. Conversion and inversion may be explained -- again -- in terms of virtual relations.

$$\ulcorner(C\underset{\sim}{R}_{\underline{n}})_{\underline{n}}\urcorner$$

is to represent

$$\ulcorner\{x_{\underline{n}}\, x_1 \ldots x_{(\underline{n}-1)} \ni \underset{\sim}{R}_{\underline{n}}\, x_1 \ldots x_{\underline{n}}\}\urcorner,$$

and

$$\ulcorner(I\underset{\sim}{R}_{\underline{n}})_{\underline{n}}\urcorner$$

is to represent

$$\ulcorner\{\underline{x}_2\ \underline{x}_1\ \underline{x}_3 \cdots \underline{x}_{\underline{n}} \ni \underline{\underline{R}}_{\underline{n}}\ \underline{x}_1 \cdots \underline{x}_{\underline{n}}\}\urcorner.$$

E functions as a combinator for forming relational domains. Thus

$$\ulcorner(E\underline{\underline{R}}_{\underline{n}})_{(\underline{n}-1)}\urcorner$$

is to represent

$$\ulcorner\{\underline{x}_1 \cdots \underline{x}_{(\underline{n}-1)} \ni (E\underline{x}_{\underline{n}})\underline{\underline{R}}_{\underline{n}}\ \underline{x}_1 \cdots \underline{x}_{\underline{n}}\}\urcorner.$$

To see how these combinators function, let us note that (1) above may now be expressed by

$$\ulcorner(E\underline{\underline{R}}_2)_1\urcorner,$$

and (9) by

$$\ulcorner-(E-\underline{\underline{R}}_2)_1\urcorner.$$

(11) above becomes

$$\ulcorner(I(C\underline{\underline{R}}_{\underline{n}})_{\underline{n}})_{\underline{n}}\urcorner,$$

(10) becomes

$$\ulcorner(E-(E(E-(E-(E-\underline{\underline{R}}_6)_5)_4)_3)_2)_1\urcorner,$$

and (12) becomes

$$\ulcorner(E(E(C(C(I-(E-(C(C(C(E(C(C(C(C\underline{\underline{R}}_6)_6)_6)_6)_5)_5)_5)_5)_4)_4)_4)_4)_4)_3)_2\urcorner,$$

and so on.

The other combinators are such that

$$\ulcorner(S(\underline{\underline{R}}_{\underline{n}},\underline{\underline{S}}_{\underline{n}}))_{\underline{n}}\urcorner,$$

$$\ulcorner NR_{n} \urcorner,$$

and

$$\ulcorner (PR_{n})_{(n+1)} \urcorner$$

represent the Boolean sums, negation, and restricted Cartesian products, respectively.

Here, as in RTL, all our formulae may be taken to be of the form $\ulcorner V_n R_n \urcorner$, where '$V_n$' is defined as there. It is easy to see that the principles Pr1-Pr15 may all be expressed as principles in the new notation.

This quasi-combinatory logic of relations is really merely RTL in a kind of notational disguise, Even so, it is just interesting enough perhaps to be given a label of its own. Let us call it 'CLR', the combinatory logic of relations. The use of the letters 'S', 'N', 'P', and so on, is akin to uses of capital roman letters in Curry's formulations. It is interesting, too, to note that CLR, like RA and RTL, can be used as a surrogate for quantification-theory. In fact, it provides a kind of combinatory theory of quantification.

Finally, it should be remarked that the theories of relations in this paper are sharply to be contrasted with the mereological theory of them discussed previously.[10] However, that theory could readily be formulated as either an RA or RTL or a CLR, where 'P' and 'Ord' are taken as the only primitive non-logical relation symbols. The mechanisms of RA or RTL or CLR would merely provide the underlying logic needed in place of the quantification theory actually used. And similarly for any other theory formulable in quantificational terms.

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FOOTNOTES

1 *"On the Algebra of Relations and their Affiliates," in* Mind, Modality, Meaning, and Method *(1983)*

2 *Recall "The Philosophic Import of Virtual Classes" in* Belief, Existence, and Meaning *(1969), and Chapters I and II of* Semiotics and Linguistic Structure *(1978)*

3 Grundlagen der Mathematik *(Springer Verlag, Berlin 1934), pp. 164 ff.*

4 *Cf. A. Church,* Introduction to Mathematical Logic *(Princeton University Press, Princeton 1956), § 42*

5 *Cf. G. Frege,* On the Foundations of Geometry and Formal Theories of Arithmetic *(Yale University Press, New Haven 1971), p. 112 and* passim

6 *Recal* Truth and Denotation *(1958), Chapter III and VIII, and* Semiotics and Linguistic Structure, *Chapter V*

7 Collected Papers, *3.45-3.149. See also "Of Lovers Servants and Benefactors", in* Peirce's Logic of Relations and Other Studies *(1979), p. 41*

8 Collected Papers, *3.328-3.403A*

9 *See especially Haskell B. Curry and Robert Feys,* Combinatory Logic I *(North-Holland Publishing Co., Amsterdam 1958), and Curry, J.R. Hindley, and J.P. Seldin,* Combinatory Logic II *(Ibid.: 1972)*

10 *See "On Mereology and the Heroic Course," in* Metaphysical Foundations: Mereology and Metalogic *(Philosophia Verlag, München 1985)*

SEMIOSIS 36 37 38

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